Table 1 Comparison of estimated parameters from estimation algorithm that includes all the modes, first and second modes only, first mode only, and none of the modes

		Estimated parameters					Computed values using Eq. (4)	
Parameters	True values	All modes included (C2)	First and second modes included (C2)	First mode included (C2)	All modes neglected (C2)	All modes neglected (C3)	(C2)	(C3)
$-C_{z_\alpha}$	2.922	2.922	2.916	2.924	2.719	2.250	2.710	1.995
		(0)	(0.21)	(0.07)	(7.0)	(22.9)[a]		
C_{z_q}	14.700	14.700	14.86	13.63	15.79	7.77	16.04	8.79
		(0)	(1.1)	(7.2)	(7.4)	(47)		
$-C_{z_\delta}$	0.435	0.435	0.420	0.579	0.392	0.059	0.252	−0.363
		(0)	(3.4)	(33)	(9.8)	(86)		
$-C_{m_\alpha}$	1.660	1.660	1.649	1.652	1.421	0.580	1.424	0.631
		(0)	(0.67)	(0.48)	(14)	(65)		
$-C_{m_q}$	34.750	34.750	34.68	34.77	31.82	26.21	33.24	28.17
		(0)	(0.20)	(0.06)	(8.4)	(25)		
$-C_{m_\delta}$	2.578	2.578	2.558	2.554	2.349	1.595	2.375	1.688
		(0)	(0.78)	(0.93)	(8.8)	(38)		

[a]Percentage of error with respect to true values.

elastic effects and is quite inadequate to yield any useful estimates as the flexibility of the aircraft increases.

It is of interest to see how the above estimated parameters for case 4 compare with the analytically based values computed using the proposed Eq. (4) for approximating such equivalent estimated parameters that would result from the use of a rigid body model in estimation algorithm. Such computed values for C2 and C3 configurations are shown in columns 8 and 9 of Table 1. Except for C_{z_δ}, the agreement in the estimated and computed values is good, with marginal deterioration for configuration C3 as compared to C2.

The above observations suggest that the proposed Eq. (4) may be useful in two different ways.

1) If the rigid body model of a flexible aircraft were tested in a wind tunnel to obtain rigid body stability and control derivatives, and the theoretical values of mode shapes and in-vacuo frequencies of the aeroelastic airplane were calculated, then one could analytically compute from Eq. (4) the expected values of equivalent parameters that would be obtained from flight data of the aeroelastic aircraft if estimation algorithm were to use the rigid body model.

2) Conversely, if rigid body wind tunnel and flight test estimations (based on rigid body model in estimation algorithm) were given, and limited structural data (in-vacuo frequencies and total force coefficients $C_{z_{n1}}$, C_α^{n1}, etc.), then Eq. (4) could be used to compute terms like C_{n1}^{n1}, etc.

The function F defined in Eq. (4) seems promising to quantitatively indicate the degree of flexibility of aircraft and thereby suggest criterion for deciding adequacies or otherwise of using simple rigid body models in estimation algorithms. Work is in progress for evolving such a criterion.

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Effects of Model Scale on Flight Characteristics and Design Parameters

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Nomenclature

C_D = drag coefficient
C_L = lift coefficient
c = velocity of sound, or wing chord
D = drag
d = ratio of density, model to full scale
L = length
l = ratio of length, model to full scale
M = mass
m = ratio of mass, model to full scale
P = power
p = ratio of power, model to full scale, or rolling velocity
S = wing area
V = airspeed
W = weight
γ = flight-path angle
μ = viscosity
ρ = density

Subscripts

mod = model
f.s. = full scale

Introduction

FLYING scale models are often used in various phases of aircraft design and research. These models range from subscale prototypes to very small scale spin-tunnel models. While applicable scaling laws are well known to specialists in various fields of research,[1] the effects of scaling on a wide range of parameters involving flight characteristics, performance, and structural design do not appear to be generally

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Table 1 Variation of quantities of interest when the mass varies as different powers of the scale

Quantity of interest	Quantity held constant: m/l	m/l^2	m/l^3	m/l^4	m/l^5	Variation with d^a (all cases)
	Variation with l^a					
Speed or Mach number (assuming same speed of sound), ft/s	$l^{-1/2}$	——	$l^{1/2}$	l	$l^{3/2}$	$d^{-1/2}$
Apparent speed, l/s	$l^{-3/2}$	l^{-1}	$l^{-1/2}$	——	$l^{1/2}$	$d^{-1/2}$
Power, ft lb/s	$l^{1/2}$	l^2	$l^{3.5}$	l^5	$l^{6.5}$	$d^{-1/2}$
Reynolds number (assuming same viscosity)	$l^{1/2}$	l	$l^{3/2}$	l^2	$l^{5/2}$	$d^{1/2}$
Froude number, V^2/gL	l	$l^{1/2}$	——	$l^{-1/2}$	l^{-1}	$d^{1/2}$
Response time constant, s	$l^{1/2}$	$l^{1/2}$	$l^{1/2}$	$l^{1/2}$	$l^{1/2}$	——
Response frequency, rad/s	$l^{-1/2}$	$l^{-1/2}$	$l^{-1/2}$	$l^{-1/2}$	$l^{-1/2}$	——
Bending stress or torsional stress, lb/in.^{2}	l^{-1}	——	l	l^2	l^3	——
Centrifugal stress, lb/in.^{2}	l^{-3}	l^{-1}	l	l^3	l^5	d^{-1}
Time for longitudinal maneuver, s	$l^{1/2}$	$l^{1/2}$	$l^{1/2}$	$l^{1/2}$	$l^{1/2}$	——
Distance for longitudinal manuever, ft	——	$l^{1/2}$	l	$l^{3/2}$	l^2	$d^{-1/2}$
Time for complete circle (or fraction thereof), s	$l^{-1/2}$	——	$l^{1/2}$	l	$l^{3/2}$	$d^{-1/2}$
Radius of circle, ft	l^{-1}	——	l	l^2	l^3	d^{-1}
Time for roll maneuver, s	$l^{3/2}$	l	$l^{1/2}$	——	$l^{-1/2}$	$d^{1/2}$
Distance for roll maneuver, ft	l	l	l	l	l	——

[a]Dash indicates quantity is unaffected.

available. This article presents a simple way to examine the effects of scaling on a wide range of parameters.

The main variables considered in this analysis are the scale and mass of the vehicle and the density of the flight medium. In practice, these variables may take on any values, but in certain familiar cases, the mass varies as some power of the scale. For example, for a constant wing loading, the mass varies as the square of the scale, or for a constant density of the vehicle, the mass varies as the cube of the scale. In order to illustrate a still wider range of conditions in a systematic manner, cases are considered herein in which the mass varies as various powers of the scale ranging from the first through the fifth power.

Analysis

The branch of engineering called dimensional analysis is closely related to the subject of scaling considered in this report.[2,3] In applications to airplane design, dimensional analysis shows that certain nondimensional parameters must be kept the same for similar flow conditions on the model and on the full-scale vehicle. The most familiar of these parameters are the Reynolds number, $\rho VL/\mu$, and the Mach number, V/c. A listing of many of the nondimensional parameters applicable to various branches of physics and engineering is given in Ref. 4.

The effects of these nondimensional parameters have been studied extensively, but they are not the main subjects of interest in this note. Because of limitations of test facilities, many types of tests are made in which these nondimensional ratios cannot be kept at their full-scale values. In this note, the main emphasis is placed on the effects of scale on flight and structural parameters when the mass of the vehicle and the density of the test medium are varied arbitrarily.

The analysis used in this note in studying the effects of scale is elementary. To illustrate the method, consider the variation of power required in steady flight, either level, climbing, or descending, with model scale, mass ratio, and density ratio. Power required for climbing flight is given by:

$$P = D \cdot V + W \sin \gamma \cdot V$$
$$= W(C_D/C_L \cos \gamma^{3/2} + \sin \gamma \cos \gamma^{1/2})\sqrt{(W/S/C_L\rho/2)}$$

If the model is assumed to have the same ratio C_D/C_L, lift coefficient, and climb angle as the full-scale aircraft, the first term in parentheses is unchanged and the ratio of the powers is

$$p = \frac{[W\sqrt{(W/S/C_L\rho/2)}]_{\text{mod}}}{[W\sqrt{(W/S/C_L\rho/2)}]_{\text{f.s.}}} = \frac{(M^{3/2}/L\rho^{1/2})_{\text{mod}}}{(M^{3/2}/L\rho^{1/2})_{\text{f.s.}}} = \frac{m^{3/2}}{ld^{1/2}}$$

As an example, consider the case of a model with the same wing loading as the full-scale aircraft. Then m/l^2 = const, or $m \sim l^2$. Thus

$$p \sim (l^3/ld^{1/2}) = l^2 d^{-1/2}$$

or the power required varies as the square of the model scale and inversely as the square root of the density ratio.

Results

Using the methods described previously, each of the characteristics of interest may be determined as a function of the mass ratio, density ratio, and scale. All that is necessary is a formula for the desired quantity. Formulas for the quantities used herein may be found in standard textbooks on the subjects involved. As stated previously, the model mass is assumed to vary as a power of the scale ranging from the first through the fifth power. These results are presented in Table 1. The mass is assumed to have no dependence on the density of the test medium. Because of the regular variation assumed for the exponent giving the variation of mass with scale, the exponents in the calculated quantities also show a regular variation, so that if two columns or Table 1 are calculated, the rest may be immediately filled in.

Discussion of Results

The degree of similarity required of a model depends on the quantity being studied. If the quantity depends only on the external contour, obviously the internal structure does not need to be scaled. In studies of dynamic motions, both the external contour and inertial properties, i.e., the ratios of radii of gyration to the length, and tilt of the principal axes, must be retained. In studies of structural stresses, all details of the structure should be geometrically scaled, though often a simplified structure may be used in the model and the relations of its stresses and deflections to those of the full-scale vehicle may be calculated.

If a model is scaled geometrically down to the finest detail of the structure, and the same materials are used in the model as on the full-scale vehicle, the mass varies as l^3. Of course, models with simpler structure and suitable ballasting may also follow this relation. As shown in Table 1, in this case, with the density of the test medium held constant, the Froude number remains the same. As a result, this type of scaling is often called Froude scaling. Also, in this case, the time constants for longitudinal and lateral maneuvers have the same variation with scale. This is the only condition in which maneuvers of the vehicle for given control inputs are geometrically similar to those of the full-scale vehicle. As a result, this condition is also called dynamic similarity. Maneuvers for given control inputs have a scale proportional to the reference length. If any problems of ground clearance are being studied, such as terrain following, the scale of the terrain must be the same as that of the model. Not all quantities remain the same as those of the full-scale vehicle. For example, the speed for flight at a given lift coefficient varies as the square root of the length. The apparent speed, varying as $l^{-1/2}$, is increased. The wing stresses are reduced directly as l, a relation that is very convenient for dynamic model tests, because the model may be made with a less efficient structure or less structurally efficient materials than the full-scale vehicle. For example, spin-tunnel models may be made of materials such as balsa wood and plywood, with plenty of room left for ballast and internal mechanisms. On the other hand, this relationship works against the design of very large airplanes. A larger structural weight fraction is required on a larger airplane to retain the same wing stresses. This relationship is often called the cube-square law, implying that the weight goes up as the volume but the lift goes up as the area. This viewpoint is oversimplified, however, because the wing stresses at a given value of the load factor increase directly with the size even when the structural members are scaled geometrically. Thus, for the same design load factor, the structure of a large vehicle must be even heavier than implied by the cube-square law.

If an attempt is made to build a model that flies at a given lift coefficient at the same speed as a full-scale vehicle, the ratio of wing loadings, or m/l^2, must equal 1 (Table 1, col. 2). In order to accomplish this result, however, the model must be ballasted or made of heavier material than the full-scale vehicle. On a smaller model, the time for a longitudinal maneuver is the same, but the time for a rolling maneuver is reduced. (As the scale is reduced to zero, the model behaves as a point mass, with instantaneous rolling maneuvers.) Bending stresses at a given value of load factor remain the same as in the full-scale vehicle, but centrifugal stresses in propellers or rotors rotating with the same advance ratio vary inversely with the scale, becoming larger in a smaller vehicle.

The time constant, or response time, for any maneuver or control input in flight at a given lift coefficient is seen from Table 1 to be dependent only on the scale and independent of the mass of the model. For example, a model of a given size made of solid metal takes the same time to respond as a model made of balsa wood, although of course the heavy model is flying faster. In view of the fact that the time constant varies as the square root of the scale, a smaller model responds faster to controls, thereby requiring greater skill to perform precision maneuvers. In tests of free-flying models in NASA wind tunnels, this problem is overcome to some extent by using separate pilots to control longitudinal and lateral motions.

Examples

Langley's Aerodromes

An interesting example of scaling is the relationship between the models and the man-carrying airplane built by Samuel P. Langley.[5] Langley built and flew successfully both steam-powered and internal-combustion engine powered models of his proposed design. These models were approximately $\frac{1}{4}$ scale and were launched by catapulting from a houseboat in the Potomac River. The full-scale vehicle, however, crashed in two attempts due to wing failure during catapulting. The full-scale "Aerodrome" had approximately the same wing loading as the models, and the weight per unit area of the wing structure was also approximately the same. Examination of Table 1, col. 2, shows that the structural stresses in steady flight in the full-scale vehicle should have been the same as in the models and indeed, the approximate strength margins were shown to be the same by structural tests. The launch acceleration was approximately the same for the full-scale airplane as for the models. A possible explanation for the failure of the full-scale vehicle appears to be the effect of the additional apparent mass of the air accelerated during the catapult launching. This mass goes up as the cube of the scale, whereas the steady loads, in Langley's case, increased as the square of the scale. With a wing structure as light as Langley's, the effect of the additional apparent mass of the air can be shown to have an important effect during a catapult launch.

X-1 Drop Model

A case in which a model was built without attention to dynamic scaling, and as a result produced unexpected results, is the X-1 drop model built at the NACA Langley Laboratory in 1946 to study longitudinal control at transonic speeds. Because wind tunnels at that time were not capable of making measurements at transonic speeds and high Reynolds numbers, heavy models were dropped from an altitude of 30,000 ft, with data on the trajectory and aerodynamic forces obtained by tracking and telemetry. The model used was a quarter-scale model or the X-1 with a span of 7 ft and a weight of 1350 lb. The elevator angle required to trim was thought to be little affected by moderate rolling velocities. Therefore, the model was adjusted to roll slowly during the drop to maintain a predictable trajectory. The model reached a Mach number of 0.98 before impacting the ground, but at a Mach number of 0.75, which was below the Mach number at which severe transonic effects were expected, a violent pitching oscillation occurred, with the model oscillating for several cycles between the negative and positive stall.

The heavy weight and large inertia of the model suggested that an analysis be made of gyroscopic effects due to rolling on longitudinal stability.[6] This report prediced the problem, later called "roll coupling," which became an important consideration in the design of high-speed fighter airplanes. The full-scale X-1 did not encounter this problem. For the large mass of this model, which corresponded roughly to the condition m/l is constant, the gyroscopic pitching and yawing moment coefficients in a steady roll at a given value of helix angle would vary as l^{-2}, or would be 16 times as great on the X-1 model as on the full-scale airplane. As a result, the exaggerated effects on the model led to instability and inspired an analysis that later had applications to full-scale fighter airplanes.

Free-Flying Models for Stall and Spin Research

Spin-tunnel models have been used for many years to study spin recovery of airplanes because of the difficulty of analyzing the complex aerodynamic forces and moments in a spin. These problems arise because of the separated flow and the complex motions of an airplane in a spin. Spin-tunnel models operate at very low values of Reynolds number compared to full-scale airplanes because of the limitations of available test facilities. With the availability of radio-control and telemetering techniques, larger-scale models dropped from helicopters have been used for some tests, both to obtain larger values of Reynolds number and to allow studies of stall and spin entry. The primary scaling consideration in these models is dynamic similarity, or Froude scaling, although sometimes the model is designed to simulate the airplane in flight at a higher altitude. Comparison of these models with hobby-type models of similar size that are designed without consideration of dynamic similarity shows that, in general, hobby-type radio-controlled models of modern airplanes have much lower values of wing loading than would be required for Froude scaling, whereas the power of hobby-type models, at least of piston-engine aircraft, is usually much greater than required to simulate the climb performance characteristics of full-scale aircraft.

Many other examples of scaling could be given, but because of the limited length of this note, they cannot be discussed herein. The technique described should allow analysis of other cases.

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Continuum Sizing Design Sensitivity Analysis of Eigenvectors Using Ritz Vectors

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I. Introduction

DESIGN sensitivity analysis (DSA) of eigenvectors is important for the study of structural dynamics, e.g., to identify and optimize structural systems, and substantial efforts have been made in this area.[1,2] Recently, Wang[3,4] proposed two modified modal methods, called the explicit and implicit methods. In the explicit method, the first load-dependent Ritz vector (LDRV)[5-7] is used in the solution, while in the implicit method, the first LDRV is added to the basis vectors. Wang[4] numerically demonstrated that the implicit method is superior to the explicit method. Nevertheless, all discrete methods of DSA[1-4] require derivatives of stiffness and mass matrices.

This Note proposes a unified continuum-based sizing DSA of eigenvectors that can be obtained without differentiating stiffness and mass matrices. Fox and Kapoor's eigenvector expansion method is modified by adding LDRVs to the basis to improve the accuracy of design sensitivity of eigenvectors. LDRVs proposed by Kline[6,7] are used for a numerical example. The numerical example studied in this Note shows that adding two LDRVs to the basis vectors yields accurate sensitivity results.

In this Note, uppercase bold letters denote matrices, lowercase bold italic letters are vectors, and lowercase letters are either scalars or functions.

II. Continuum Design Sensitivity of Eigenvectors

In order to avoid differentiation of stiffness and mass matrices, the variational equations of eigenvalue problems are differentiated in the continuum design sensitivity of eigenfunctions. For a given structural system with physical domain Ω, variational equations of eigenvalue problems can be written as[8]

$$a_u(y^i, \bar{y}) = \zeta_i d_u(y^i, \bar{y}) \tag{1}$$

for all $\bar{y} \in Z$; and the orthonormalizing condition employed is

$$d_u(y^i, y^j) = d_u(\psi^i, \psi^j) = d_u(\phi^i, \phi^j) = \delta_{ij} \tag{2}$$

where Z is the vector space of kinematically admissible displacements, $a_u(\ )$ is the strain energy bilinear form, $d_u(\ )$ is the mass effect bilinear form, $\bar{y}(\boldsymbol{x})$ is the kinematically admissible eigenfunction with $\boldsymbol{x} \in \Omega$, $y(\boldsymbol{x})$ is the eigenfunction, ζ is the eigenvalue, $\psi(\boldsymbol{x})$ is the Ritz function, ϕ is the basis function, i.e., either the eigenfunction or Ritz function, and δ_{ij} is the Kronecker delta. In Eqs. (1) and (2), the subscript u denotes dependency of the bilinear forms on a design variable u.

Design sensitivity of the eigenvalue is completed first. Substituting $\bar{y} = y^j$ in Eq. (1) and using Eq. (2)

$$a_u(y^i, y^j) = \zeta_i \delta_{ij} \tag{3}$$

Taking the first variation of Eq. (1) with respect to a design variable u yields

$$a'_{\delta u}(y^i, \bar{y}) + a_u(y^{i'}, \bar{y}) = \zeta'_i d_u(y^i, \bar{y}) + \zeta_i d'_{\delta u}(y^i, \bar{y}) + \zeta_i d_u(y^{i'}, \bar{y}) \tag{4}$$

which must hold for all $\bar{y} \in Z$. Letting $\bar{y} = y^i$ in Eq. (4) and using Eq. (2)

$$\zeta'_i = a'_{\delta u}(y^i, y^i) - \zeta_i d'_{\delta u}(y^i, y^i) + [a_u(y^{i'}, y^i) - \zeta_i d_u(y^{i'}, y^i)] \tag{5}$$

Since $y^{i'}$ satisfies the same kinematic boundary conditions as y^i, thus $y^{i'} \in Z$.[8] Using Eq. (1), the last two terms of Eq. (5) cancel each other, and design sensitivity of the ith eigenvalue ζ_i is

$$\zeta'_i = a'_{\delta u}(y^i, y^i) - \zeta_i d'_{\delta u}(y^i, y^i) \tag{6}$$

Rearranging Eq. (4), the continuum equation for the design sensitivity of the ith eigenfunction y^i is obtained as

$$a_u(y^{i'}, \bar{y}) - \zeta_i d_u(y^{i'}, \bar{y}) = -a'_{\delta u}(y^i, \bar{y}) + \zeta_i d'_{\delta u}(y^i, \bar{y}) + \zeta'_i d_u(y^i, \bar{y}) \tag{7}$$

which must hold for all $\bar{y} \in Z$.

III. Approximate Design Sensitivity Analysis

The design sensitivity of the ith eigenfunction y^i with respect to a design variable u is approximated by

$$y^{i'} = \boldsymbol{\phi c} \tag{8}$$

where $\boldsymbol{\phi}$ is a vector function such that $\boldsymbol{\phi} = [\boldsymbol{y}(\boldsymbol{x})^T, \boldsymbol{\psi}(\boldsymbol{x})^T]^T = [y^1(\boldsymbol{x}) \ldots y^q(\boldsymbol{x}), \psi^1(\boldsymbol{x}) \ldots \psi^{r-q}(\boldsymbol{x})]^T$. Substituting Eq. (8) into Eq. (7) and letting $\bar{y} = \phi^j$, $k = 1, 2, \ldots, r$, in Eq. (7), and $r \times r$ matrix equation is obtained as

$$[a_u(\phi^k, \phi^j) - \zeta_i d_u(\phi^k, \phi^j)]\boldsymbol{c} = [-a'_{\delta u}(y^i, \phi^j) + \zeta_i d'_{\delta u}(y^i, \phi^j) + \zeta'_i d_u(y^i, \phi^j)] \tag{9}$$

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